ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 331 (2007) 669–685
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Finite element error analysis of a zeroth order approximate deconvolution model based on a mixed formulation

Carolina Cardoso Manica [a,1], Songul Kaya Merdan [b,*]

[a] *Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA*
[b] *Department of Mathematics, Middle East Technical University, Ankara 06531, Turkey*

Received 11 April 2006
Available online 4 October 2006
Submitted by V.J. Ervin

Abstract

A suitable discretization for the Zeroth Order Model in Large Eddy Simulation of turbulent flows is sought. This is a low order model, but its importance lies in the insight that it provides for the analysis of the higher order models actually used in practice by the pioneers Stolz and Adams [N.A. Adams, S. Stolz, On the approximate deconvolution procedure for LES, Phys. Fluids 2 (1999) 1699–1701; N.A. Adams, S. Stolz, Deconvolution methods for subgrid-scale approximation in large eddy simulation, in: B.J. Geurts (Ed.), Modern Simul. Strategies for Turbulent Flow, Edwards, Philadelphia, 2001, pp. 21–44] and others. The higher order models have proven to be of high accuracy. However, stable discretizations of them have proven to be tricky and other stabilizations, such as time relaxation and eddy viscosity, are often added. We propose a discretization based on a mixed variational formulation that gives the correct energy balance. We show it to be unconditionally stable and prove convergence.

Keywords: Zeroth Order Model; Deconvolution; Approximate Deconvolution Models; Large Eddy Simulation

* Corresponding author.
E-mail addresses: cac15@pitt.edu (C.C. Manica), smerdan@metu.edu.tr (S.K. Merdan).
URLs: http://www.pitt.edu/~cac15 (C.C. Manica), http://www.metu.edu.tr/~smerdan (S.K. Merdan).
1 Partially supported by NSF grants DMS 0207627 and DMS 0508260.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.08.083

1. Introduction

In this report, we consider a new discretization of the Approximate Deconvolution Models in Large Eddy Simulation (LES), focusing on the pivotal zeroth order model. The usual finite element approach was already considered in [15] and its stability was proven to be dependent on the exact way the filtering operation is performed. The discretization we propose here grows out of the natural formulation for the continuous model, i.e. it comes directly from a formulation that gives the correct energy balance for the large scales. It is inspired by the technique used in [12] to prove existence and uniqueness of strong solutions in the continuous case. In contrast to the approach of [15], it is less sensitive to the details of the filter, but its implementation introduces more degrees of freedom. After all, Ferziger [5], "... there is a close connection between the numerical methods and the modeling approach used in simulation; this connection has not been sufficiently appreciated ...." We prove that the new discretization is stable and give optimal convergence rates, including an analysis of time averaged errors.

We are interested in designing a numerical method for approximating flow averages of flows at higher Reynolds number subject to the no-slip boundary condition, described by the incompressible Navier–Stokes equations (NSE)

$$
\begin{aligned}
&\mathbf{u}_t + \nabla\cdot(\mathbf{u}\mathbf{u}) - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} && \text{in } (0,T]\times\Omega,\\
&\nabla\cdot\mathbf{u} = 0 && \text{in } [0,T]\times\Omega,\\
&\mathbf{u} = \mathbf{0} && \text{in } [0,T]\times\partial\Omega,\\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega,\\
&\int_\Omega p\,d\mathbf{x} = 0,
\end{aligned}
\tag{1.1}
$$

where $\mathbf{u}$ is the fluid velocity, p the pressure, $\mathbf{f}$ the external force, $\nu > 0$ the kinematic viscosity, and $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, a bounded, simply connected domain with polygonal boundary $\partial\Omega$.

Let overbar denote a spacial averaging operator which preserves the no-slip condition and let $\delta > 0$ denote the averaging radius. Following [13], we define the filtering operation as the solution of a shifted Poisson problem. The differential filter $\bar{\boldsymbol{\phi}}$ of a quantity $\boldsymbol{\phi}$ is the solution to the boundary value problem

$$
\begin{aligned}
&-\delta^2\Delta\bar{\boldsymbol{\phi}} + \bar{\boldsymbol{\phi}} = \boldsymbol{\phi} && \text{in } \Omega,\\
&\bar{\boldsymbol{\phi}} = 0 && \text{on } \partial\Omega.
\end{aligned}
\tag{1.2}
$$

Applying this filtering operation to the Navier–Stokes equations (1.1) results

$$
\begin{aligned}
&\bar{\mathbf{u}}_t + \overline{\nabla\cdot(\mathbf{u}\mathbf{u})} - \nu\overline{\Delta\mathbf{u}} + \overline{\nabla p} = \mathbf{f} && \text{in } (0,T]\times\Omega,\\
&\overline{\nabla\cdot\mathbf{u}} = 0 && \text{in } [0,T]\times\Omega,\\
&\bar{\mathbf{u}} = \mathbf{0} && \text{in } [0,T]\times\partial\Omega,\\
&\bar{\mathbf{u}}(0,\mathbf{x}) = \bar{\mathbf{u}}_0(\mathbf{x}) && \text{in } \Omega,\\
&\int_\Omega p\,d\mathbf{x} = 0.
\end{aligned}
\tag{1.3}
$$

The system of equations (1.3) is not closed, suggesting that $\mathbf{u}$ must be modeled in terms of $\bar{\mathbf{u}}$. We choose here the simplest closure model, $\mathbf{u} \simeq \bar{\mathbf{u}}(+\mathcal{O}(\delta^2))$, known as the Zeroth Order Model because it is exact on constant flows. Since we are interested in nonperiodic flow, an $\mathcal{O}(\delta^2)$ commutation error is introduced in the incompressibility constraint. Letting $\mathbf{w}$ be an approximation to $\bar{\mathbf{u}}$, and imposing $\nabla \cdot \mathbf{w} = 0$, system (1.3) becomes

$$\begin{aligned}
&\mathbf{w}_t - \nu\Delta\mathbf{w} + \overline{\nabla \cdot (\mathbf{w}\mathbf{w})} + \overline{\nabla p} = \bar{\mathbf{f}} && \text{in } (0,T] \times \Omega,\\
&\nabla \cdot \mathbf{w} = 0 && \text{in } [0,T] \times \Omega,\\
&\mathbf{w} = \mathbf{0} && \text{in } [0,T] \times \partial\Omega,\\
&\mathbf{w}(0,\mathbf{x}) = \bar{\mathbf{u}}_0(\mathbf{x}) && \text{in } \Omega.
\end{aligned} \tag{1.4}$$

This model has been extensively studied from an analytical point of view in the case of periodic boundary conditions. In [11] it is shown that the model is stable and has weak solutions. In [12] it was proven that strong solutions exist and are unique, the modeling error was bounded and convergence as $\delta \to 0$ to a solution of the NSE is proven. The Zeroth Order Model is the simplest model in the family of Approximate Deconvolution Models (ADM) introduced by Stolz and Adams [1,2]. Despite being a low order model, it is the key in understanding mathematically how the higher order models in the family behave. The next step is to extend the techniques used herein to the other ADM. The methods used in [11,12] were extended for the whole family in [4] to prove an energy inequality, existence, uniqueness and regularity of strong solutions and also to give a rigorous bound on the modeling error. Since their modeling error is $\mathcal{O}(\delta^{2N+2})$, for $N = 1, 2, 3, \ldots$, one would have to introduce a filter that preserves incompressibility, such as the Stokes filter.

Another remarkable property of this family of models, including (1.4), is that their time averaged consistency error converges to zero uniformly in the Reynolds number as $\mathcal{O}(\delta^{1/3})$ [14].

Here, we were inspired by the idea in [12], in which the variational formulation is in $H^2(\Omega)$ (see Section 3). This would be computationally expensive, requiring the use of C^1 elements. Instead, we study a mixed formulation that requires less regularity of the true solution $\mathbf{w}$. The error analysis is performed and optimal convergence rates are derived. We also include a section on time averaged errors, since this method is designed for simulation of turbulent flows. In such cases, the usual procedure is to compute time averages of the physical quantities of interest [3,16].

The report is organized as follows. In Section 2, we introduce notation and give some preliminaries. The derivation of the discretization and its stability properties are explained in Section 3. Optimal convergence rates are derived in Section 4, with the help of a modified Stokes projection. In Section 5, time averaged errors are analyzed and finally, some conclusions and remarks are presented in Section 6.

2. Notation and preliminaries

We now introduce the notation for the functional settings. The inner product and norm in $(L^2(\Omega))^d$, $d = 2, 3$, are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$. The norm in $(H^k(\Omega))^d$ is denoted by $\|\cdot\|_k$ and the norms in Lebesgue spaces $(L^p(\Omega))^d$, $1 \leqslant p < \infty$, $p \neq 2$, by $\|\cdot\|_{L^p}$. The velocity and pressure spaces are $\mathbf{X} = (H_0^1(\Omega))^d$ and $Q = L_0^2(\Omega)$, respectively. For $\mathbf{f}$ an element in the dual space of $\mathbf{X}$, its norm is defined by

$$\|\mathbf{f}\|_{-1} = \sup_{\mathbf{v}\in\mathbf{X}} \frac{|(\mathbf{f},\mathbf{v})|}{\|\nabla\mathbf{v}\|}.$$

The space of weakly divergence free functions is defined as

$$\mathbf{V} = \big\{\mathbf{v} \in \mathbf{X}:\ (q, \nabla \cdot \mathbf{v}) = 0,\ \forall q \in Q\big\}.$$

The following trilinear form,

$$b(\mathbf{w},\mathbf{u},\mathbf{v})=\frac{1}{2}\big(\big((\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{v}\big)-\big((\mathbf{w}\cdot\nabla)\mathbf{v},\mathbf{u}\big)\big),$$

is the skew-symmetric form of the convective term. We often use the following properties:

$$b(\mathbf{u},\mathbf{v},\mathbf{w})=-b(\mathbf{u},\mathbf{w},\mathbf{v})\quad\text{and}\quad b(\mathbf{u},\mathbf{v},\mathbf{v})=0,\quad\forall\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}.$$

Furthermore (see [6] for a proof), if $\Omega\subset\mathbb{R}^d$, $d=2,3$, there exists a constant $M=M(\Omega)<\infty$ such that

$$b(\mathbf{u},\mathbf{v},\mathbf{w})\leqslant M\|\nabla\mathbf{u}\|\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|,\quad\forall\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}.\tag{2.1}$$

Particularly, when $d=3$, this can be improved to

$$b(\mathbf{u},\mathbf{v},\mathbf{w})\leqslant M\sqrt{\|\mathbf{u}\|\|\nabla\mathbf{u}\|}\,\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|,\quad\forall\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}.\tag{2.2}$$

The discrete analogue of the model begins with constructing conforming finite element spaces $\mathbf{X}^h\subset\mathbf{X}$, $Q^h\subset Q$ where h denotes mesh width for $(\mathbf{X}^h,Q^h)$. These spaces satisfy the usual approximation theoretic conditions and the inf-sup condition or Babuska–Brezzi condition, i.e. there is a constant β independent of mesh size h such that

$$\inf_{q^h\in Q^h}\sup_{\mathbf{v}^h\in\mathbf{X}^h}\frac{(q^h,\nabla\cdot\mathbf{v}^h)}{\|\nabla\mathbf{v}^h\|\|q^h\|}\geqslant\beta>0.\tag{2.3}$$

For examples of such compatible spaces see, e.g., Gunzburger [7], Girault and Raviart [6]. The space of discretely divergence free functions is defined by

$$\mathbf{V}^h=\big\{\mathbf{v}\in\mathbf{X}^h\colon\ \big(\nabla\cdot\mathbf{v}^h,q^h\big)=0,\ \forall q^h\in Q^h\big\},$$

which is a nontrivial closed subspace of $\mathbf{X}^h$ under the inf-sup condition (2.3). It is known that even if typically $\mathbf{V}^h\nsubseteq\mathbf{V}$, under (2.3), the functions in $\mathbf{V}$ are well approximated by ones in $\mathbf{V}^h$ [6,7].

In the analysis, we often use the following inequalities:
Young's Inequality:

$$ab\leqslant\frac{\epsilon}{q}a^q+\frac{\epsilon^{-q/p}}{p}b^p,\quad 1<q,p<\infty,\ \frac{1}{q}+\frac{1}{p}=1.$$

Poincaré's Inequality:

$$\|\mathbf{v}\|\leqslant C_P\|\nabla\mathbf{v}\|,\quad\forall\mathbf{v}\in\mathbf{X},$$

where C_P is a constant depending on Ω.

Time averages are denoted by $\langle\cdot\rangle$; for example, the time average of $\boldsymbol{\psi}$ is

$$\langle\boldsymbol{\psi}\rangle=\limsup_{T\to\infty}\frac{1}{T}\int_0^T\boldsymbol{\psi}(t)\,dt.$$

Time averages satisfy a Cauchy–Schwarz type of inequality [10]:

$$\big\langle(\boldsymbol{\psi},\boldsymbol{\chi})\big\rangle\leqslant\big\langle\|\boldsymbol{\psi}\|^2\big\rangle^{1/2}\big\langle\|\boldsymbol{\chi}\|^2\big\rangle^{1/2}.$$

3. Derivation of the new discretization

This section develops a mixed variational formulation for (1.4) and its finite element discretization. We recall that the operations of differentiation and filtering do not commute and use a strategy that gives the correct balance of energy for the model. The stability of the discrete solution is also investigated.

By choosing a differential filter as an averaging operator, following the discussion in [12], we define $A\mathbf{v} = -\delta^2\Delta\mathbf{v} + \mathbf{v}$, for all $\mathbf{v} \in \mathbf{X} \cap (H^2(\Omega))^d$, so that $A\bar{\boldsymbol{\phi}} = \boldsymbol{\phi}$. Note that since the Laplace operator Δ is self-adjoint, so is A.

Let $\mathbf{w}$ be a smooth strong solution of (1.4). The development of the model starts with multiplying (1.4) by $A\mathbf{v}$ and integrating over the domain. One has

$$(\mathbf{w}_t, A\mathbf{v}) - \nu(\Delta\mathbf{w}, A\mathbf{v}) + \big(\overline{\nabla \cdot (\mathbf{w}\mathbf{w})}, A\mathbf{v}\big) + (\overline{\nabla p}, A\mathbf{v}) = (\bar{\mathbf{f}}, A\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{X} \cap \big(H^2(\Omega)\big)^d.$$

By using the self-adjointness of the operator A together with property (1.2), followed by integration by parts, we derive the following variational formulation: Find $\mathbf{w} : [0, T] \to \mathbf{X} \cap (H^2(\Omega))^d$, $p : (0, T] \to Q$ satisfying $\mathbf{w}(0, \mathbf{x}) = \bar{\mathbf{u}}_0(\mathbf{x})$ and

$$\begin{aligned}
&(\mathbf{w}_t, \mathbf{v}) + \delta^2(\nabla\mathbf{w}_t, \nabla\mathbf{v}) + \nu\big[(\nabla\mathbf{w}, \nabla\mathbf{v}) + \delta^2(\Delta\mathbf{w}, \Delta\mathbf{v})\big] + \big(\nabla \cdot (\mathbf{w}\mathbf{w}), \mathbf{v}\big) \\
&\quad - (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \\
&(\nabla \cdot \mathbf{w}, q) = 0, \qquad (3.1)
\end{aligned}$$

for all $(\mathbf{v}, q) \in (\mathbf{X} \cap (H^2(\Omega))^d, Q)$.

No similar formulation follows by the usual approach of multiplying by $\mathbf{v}$ and integrating by parts. The formulation (3.1) contains the term $(\Delta\mathbf{w}, \Delta\mathbf{v})$ which is a fourth order term. This suggests using C^1 elements. Instead, we consider a mixed formulation of (3.1): Find $\mathbf{w} : [0, T] \to \mathbf{X}$, $\boldsymbol{\phi} : [0, T] \to \mathbf{X}$ and $p : (0, T] \to Q$ satisfying $\mathbf{w}(0, \mathbf{x}) = \bar{\mathbf{u}}_0(\mathbf{x})$ and

$$\begin{aligned}
&(\mathbf{w}_t, \mathbf{v}) + \delta^2(\nabla\mathbf{w}_t, \nabla\mathbf{v}) + b(\mathbf{w}, \mathbf{w}, \mathbf{v}) + \nu(\nabla\mathbf{w}, \nabla\mathbf{v}) + \nu\delta^2(\nabla\boldsymbol{\phi}, \nabla\mathbf{v}) \\
&\quad - (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \qquad (3.2) \\
&(\nabla\mathbf{w}, \nabla\boldsymbol{\xi}) = (\boldsymbol{\phi}, \boldsymbol{\xi}), \qquad (3.3) \\
&(\nabla \cdot \mathbf{w}, q) = 0, \qquad (3.4)
\end{aligned}$$

for all $(\mathbf{v}, \boldsymbol{\xi}, q) \in (\mathbf{X}, \mathbf{X}, Q)$.

In $\mathbf{V}$, this formulation becomes: Find $(\mathbf{w}, \boldsymbol{\phi}) : [0, T] \to (\mathbf{V}, \mathbf{X})$ such that

$$(\mathbf{w}_t, \mathbf{v}) + \delta^2(\nabla\mathbf{w}_t, \nabla\mathbf{v}) + b(\mathbf{w}, \mathbf{w}, \mathbf{v}) + \nu(\nabla\mathbf{w}, \nabla\mathbf{v}) + \nu\delta^2(\nabla\boldsymbol{\phi}, \nabla\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \qquad (3.5)$$

$$(\nabla\mathbf{w}, \nabla\boldsymbol{\xi}) = (\boldsymbol{\phi}, \boldsymbol{\xi}), \qquad (3.6)$$

for all $(\mathbf{v}, \boldsymbol{\xi}) \in (\mathbf{V}, \mathbf{X})$.

The kinetic energy and the energy dissipation rate at time t associated with this model are defined as

$$\kappa(\mathbf{w}) = \frac{1}{2}\big(\|\mathbf{w}(t)\|^2 + \delta^2\|\nabla\mathbf{w}(t)\|^2\big) \quad \text{and} \quad \varepsilon(\mathbf{w}, \boldsymbol{\phi}) = \frac{\nu}{|\Omega|}\big(\|\nabla\mathbf{w}(t)\|^2 + \delta^2\|\boldsymbol{\phi}(t)\|^2\big),$$

where $|\Omega|$ is the measure of Ω.

We first establish uniformly boundedness of the kinetic energy of $\mathbf{w}$ at time T.

Lemma 3.1. *Let* $\mathbf{f} \in L^\infty(0,\infty; H^{-1}(\Omega))$. *Then the kinetic energy* $\kappa(\mathbf{w})$ *at time* T *is uniformly bounded as*

$$\kappa(\mathbf{w}) \leqslant \left(\left\|\mathbf{w}(0)\right\|^2 + \delta^2\left\|\nabla\mathbf{w}(0)\right\|^2\right)e^{-\nu C_P^{-2}T} + \nu^{-2}C_P^2\|\mathbf{f}\|^2_{L^\infty(0,\infty;H^{-1}(\Omega))}.$$

In particular,

$$\lim_{T\to\infty}\frac{1}{T}\kappa(\mathbf{w}) = 0.$$

Proof. Set $\mathbf{v} = \mathbf{w}$ in (3.5) and $\boldsymbol{\xi} = \boldsymbol{\phi}$ in (3.6). Then, since $b(\mathbf{w},\mathbf{w},\mathbf{w}) = 0$, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{w}\|^2 + \delta^2\|\nabla\mathbf{w}\|^2\right) + \frac{\nu}{2}\left(\|\nabla\mathbf{w}\|^2 + 2\delta^2\|\boldsymbol{\phi}\|^2\right) \leqslant \frac{1}{2\nu}\|\mathbf{f}\|^2_{-1}. \tag{3.7}$$

Letting $\xi = \mathbf{w}$ in Eq. (3.6) and using Poincaré's inequality, we have that $\|\nabla\mathbf{w}\| \leqslant C_P\|\boldsymbol{\phi}\|$, then (3.7) becomes

$$\frac{d}{dt}\left(\|\mathbf{w}\|^2 + \delta^2\|\nabla\mathbf{w}\|^2\right) + \nu C_P^{-2}\left(\|\mathbf{w}\|^2 + \delta^2\|\nabla\mathbf{w}\|^2\right) \leqslant \nu^{-1}\|\mathbf{f}\|^2_{-1}. \tag{3.8}$$

Setting $y = \|\mathbf{w}\|^2 + \delta^2\|\nabla\mathbf{w}\|^2$ and using an integrating factor, Eq. (3.8) gives

$$y(T) \leqslant y(0)e^{-\nu C_P^{-2}T} + \nu^{-2}C_P^2\|\mathbf{f}\|^2_{L^\infty(0,T;H^{-1}(\Omega))}.$$

This proves the uniform boundedness. Now, dividing by T and taking the limit as $T \to \infty$ gives the second claim. □

Remark 3.1. One can also show that the total energy is bounded. We only present the proof for the discrete case (Lemma 3.3), since the idea is the same in both cases.

Our goal is to understand the behavior of numerical methods based on (3.2)–(3.4). Therefore, we consider a continuous in time finite element discretization of the problem (3.2)–(3.4). Let $\mathbf{X}^h \subset \mathbf{X}$ and $Q^h \subset Q$ satisfy (2.3). The finite element approximation to $(\mathbf{w},\boldsymbol{\phi},p)$ are maps $\mathbf{w}^h : [0,T] \to \mathbf{X}^h$, $\boldsymbol{\phi}^h : [0,T] \to \mathbf{X}^h$ and $p^h : (0,T] \to Q^h$ such that

$$\begin{aligned}
&\left(\mathbf{w}_t^h,\mathbf{v}^h\right) + \delta^2\left(\nabla\mathbf{w}_t^h,\nabla\mathbf{v}^h\right) + b\left(\mathbf{w}^h,\mathbf{w}^h,\mathbf{v}^h\right) + \nu\left(\nabla\mathbf{w}^h,\nabla\mathbf{v}^h\right) + \nu\delta^2\left(\nabla\boldsymbol{\phi}^h,\nabla\mathbf{v}^h\right)\\
&\quad - \left(p^h,\nabla\cdot\mathbf{v}^h\right) = \left(\mathbf{f},\mathbf{v}^h\right), &(3.9)\\
&\left(\nabla\mathbf{w}^h,\nabla\boldsymbol{\xi}^h\right) = \left(\boldsymbol{\phi}^h,\boldsymbol{\xi}^h\right), &(3.10)\\
&\left(q^h,\nabla\cdot\mathbf{w}^h\right) = 0, &(3.11)
\end{aligned}$$

for all $(\mathbf{v}^h,\boldsymbol{\xi}^h,q^h) \in (\mathbf{X}^h,\mathbf{X}^h,Q^h)$.

In $\mathbf{V}^h$, the semi-discrete approximation of (3.9)–(3.11) is: Find $(\mathbf{w}^h,\boldsymbol{\phi}^h) \in (\mathbf{V}^h,\mathbf{X}^h)$ such that

$$\begin{aligned}
&\left(\mathbf{w}_t^h,\mathbf{v}^h\right) + \delta^2\left(\nabla\mathbf{w}_t^h,\nabla\mathbf{v}^h\right) + b\left(\mathbf{w}^h,\mathbf{w}^h,\mathbf{v}^h\right) + \nu\left(\nabla\mathbf{w}^h,\nabla\mathbf{v}^h\right)\\
&\quad + \nu\delta^2\left(\nabla\boldsymbol{\phi}^h,\nabla\mathbf{v}^h\right) = \left(\mathbf{f},\mathbf{v}^h\right), &(3.12)\\
&\left(\nabla\mathbf{w}^h,\nabla\boldsymbol{\xi}^h\right) = \left(\boldsymbol{\phi}^h,\boldsymbol{\xi}^h\right), &(3.13)
\end{aligned}$$

for all $(\mathbf{v}^h,\boldsymbol{\xi}^h) \in (\mathbf{V}^h,\mathbf{X}^h)$.

A discrete version of Lemma 3.1 shows that the kinetic energy of the discrete solution is also uniformly bounded.

Lemma 3.2. *Let* $\mathbf{f} \in L^{\infty}(0, \infty; H^{-1}(\Omega))$. *Then the kinetic energy* $\kappa(\mathbf{w}^h)$ *is uniformly bounded as*

$$\kappa(\mathbf{w}^h) \leqslant \left(\|\mathbf{w}^h(0)\|^2 + \delta^2\|\nabla \mathbf{w}^h(0)\|^2\right)e^{-\nu C_P^{-2}T} + \nu^{-2}C_P^2\|\mathbf{f}\|^2_{L^{\infty}(0,\infty;H^{-1}(\Omega))}.$$

As a consequence,

$$\lim_{T\to\infty}\frac{1}{T}\kappa(\mathbf{w}^h) = 0.$$

Proof. The claim exactly follows as in the continuous case of Lemma 3.1. □

In addition, the next result shows that the total energy of the approximate solution $\mathbf{w}^h$ is bounded.

Lemma 3.3 *(Stability of* $\mathbf{w}^h$*). Let* $\mathbf{f} \in L^2(0, T; H^{-1}(\Omega))$. *Then any solution* $(\mathbf{w}^h, \boldsymbol{\phi}^h)$ *of* (3.12)–(3.13) *satisfies the following stability bound*:

$$\frac{1}{2}\|\mathbf{w}^h(t)\|^2 + \frac{\delta^2}{2}\|\nabla \mathbf{w}^h(t)\|^2 + \int_0^t \left[\frac{\nu}{2}\|\nabla \mathbf{w}^h\|^2 + \nu\delta^2\|\boldsymbol{\phi}^h\|^2\right]dt'$$
$$\leqslant \frac{1}{2}\|\mathbf{w}^h(0)\|^2 + \frac{\delta^2}{2}\|\nabla \mathbf{w}^h(0)\|^2 + \frac{1}{2\nu}\int_0^t \|\mathbf{f}\|^2_{-1}\,dt'.$$

Proof. Set $\mathbf{v}^h = \mathbf{w}^h$ in (3.12), $\boldsymbol{\xi}^h = \boldsymbol{\phi}^h$ in (3.13) and use $b(\mathbf{w}^h, \mathbf{w}^h, \mathbf{w}^h) = 0$, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{w}^h\|^2 + \delta^2\|\nabla \mathbf{w}^h\|^2\right) + \nu\|\nabla \mathbf{w}^h\|^2 + \nu\delta^2\left(\nabla\boldsymbol{\phi}^h, \nabla \mathbf{w}^h\right) = \left(\mathbf{f}, \mathbf{w}^h\right), \tag{3.14}$$
$$\left(\nabla \mathbf{w}^h, \nabla\boldsymbol{\phi}^h\right) = \left(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h\right). \tag{3.15}$$

Multiply (3.15) by $\nu\delta^2$ and add to (3.14) and use Cauchy–Schwarz inequality. One has

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{w}^h\|^2 + \delta^2\|\nabla \mathbf{w}^h\|^2\right) + \frac{\nu}{2}\|\nabla \mathbf{w}^h\|^2 + \nu\delta^2\|\boldsymbol{\phi}^h\|^2 \leqslant \frac{1}{2\nu}\|\mathbf{f}\|^2_{-1}.$$

Integrating the last equation over $(0, t)$ with $t \leqslant T$ gives the required result. □

4. Convergence analysis

It is useful to define the following modified Stokes projection aiming at simplifying the error analysis.

Definition 4.1 *(Modified Stokes projection).* The modified Stokes projection operator $P_S : (\mathbf{X}, \mathbf{X}, Q) \to (\mathbf{X}^h, \mathbf{X}^h, Q^h)$ is defined as follows: Let $P_S(\mathbf{w}, \boldsymbol{\phi}, p) = (\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}}, \tilde{p})$ where $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}}, \tilde{p})$ satisfies

$$\nu\left(\nabla(\mathbf{w} - \tilde{\mathbf{w}}), \nabla \mathbf{v}^h\right) + \nu\delta^2\left(\nabla(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}), \nabla \mathbf{v}^h\right) - \left(p - \tilde{p}, \nabla\cdot \mathbf{v}^h\right) = 0,$$
$$\left(\nabla(\mathbf{w} - \tilde{\mathbf{w}}), \nabla\boldsymbol{\xi}^h\right) = \left(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}, \boldsymbol{\xi}^h\right),$$
$$\left(q^h, \nabla\cdot(\mathbf{w} - \tilde{\mathbf{w}})\right) = 0, \tag{4.1}$$

for all $(\mathbf{v}^h, \boldsymbol{\xi}^h, q^h) \in (\mathbf{X}^h, \mathbf{X}^h, Q^h)$.

In $(\mathbf{V}^h, \mathbf{X}^h)$, this formulation reads: Given $(\mathbf{w}, \boldsymbol{\phi})$, find $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}}) \in (\mathbf{V}^h, \mathbf{X}^h)$ satisfying

$$\nu\big(\nabla(\mathbf{w}-\tilde{\mathbf{w}}), \nabla\mathbf{v}^h\big) + \nu\delta^2\big(\nabla(\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}), \nabla\mathbf{v}^h\big) - \big(p-q^h, \nabla\cdot\mathbf{v}^h\big) = 0, \tag{4.2}$$

$$\big(\nabla(\mathbf{w}-\tilde{\mathbf{w}}), \nabla\boldsymbol{\xi}^h\big) = \big(\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}, \boldsymbol{\xi}^h\big), \tag{4.3}$$

for all $(\mathbf{v}^h, \boldsymbol{\xi}^h) \in (\mathbf{V}^h, \mathbf{X}^h)$ and any $q^h \in Q^h$.

Under the discrete inf-sup condition (2.3), $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}}, \tilde{p})$ is a quasi optimal approximation of $(\mathbf{w}, \boldsymbol{\phi}, p)$. Since the stability and error estimation of the projection operator will be used to approximate the error between $\mathbf{w}$ and $\mathbf{w}^h$, we now give two related results.

Proposition 4.1 *(Stability of the modified Stokes projection). Let $(\mathbf{w}, \boldsymbol{\phi})$ be given. Then, $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}})$ satisfies*

$$\nu\|\nabla\tilde{\mathbf{w}}\|^2 + \nu\delta^2\|\tilde{\boldsymbol{\phi}}\|^2 \leqslant C\big\{\big(\nu+\nu\delta^2h^{-2}\big)\|\nabla\mathbf{w}\|^2 + \nu\delta^4\|\nabla\boldsymbol{\phi}\|^2 + \nu\delta^2\|\boldsymbol{\phi}\|^2 + \nu^{-1}\|p\|^2\big\}$$

and

$$\|\tilde{p}\| \leqslant C\big\{\|p\| + \nu\|\nabla\mathbf{w}\| + \nu\delta^2\|\nabla\boldsymbol{\phi}\| + \nu\|\nabla\tilde{\mathbf{w}}\| + \nu\delta^2\|\tilde{\boldsymbol{\phi}}\|\big\},$$

where C is independent of ν, δ and h.

Proof. We first set $\mathbf{v}^h = \tilde{\mathbf{w}}$ in (4.2) and $\boldsymbol{\xi}^h = \tilde{\boldsymbol{\phi}}$ in (4.3). Then, we obtain

$$\nu\|\nabla\tilde{\mathbf{w}}\|^2 = \nu(\nabla\mathbf{w}, \nabla\tilde{\mathbf{w}}) + \nu\delta^2\big(\nabla(\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}), \nabla\tilde{\mathbf{w}}\big) - (p, \nabla\cdot\tilde{\mathbf{w}}), \tag{4.4}$$

$$(\nabla\tilde{\mathbf{w}}, \nabla\tilde{\boldsymbol{\phi}}) = (\nabla\mathbf{w}, \nabla\tilde{\boldsymbol{\phi}}) + (\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}}). \tag{4.5}$$

Multiplying (4.5) by $\nu\delta^2$, substituting in (4.4) and applying the Cauchy–Schwarz inequality yields

$$\begin{aligned}\nu\|\nabla\tilde{\mathbf{w}}\|^2 + \nu\delta^2\|\tilde{\boldsymbol{\phi}}\|^2 \leqslant{}& \nu\|\nabla\mathbf{w}\|\|\nabla\tilde{\mathbf{w}}\| + \nu\delta^2\|\nabla\boldsymbol{\phi}\|\|\nabla\tilde{\mathbf{w}}\| + \nu\delta^2\|\nabla\mathbf{w}\|\|\nabla\tilde{\boldsymbol{\phi}}\| \\ &+ \nu\delta^2\|\boldsymbol{\phi}\|\|\tilde{\boldsymbol{\phi}}\| + \|p\|\|\nabla\cdot\tilde{\mathbf{w}}\|.\end{aligned}$$

Lastly, we apply the following inverse inequality

$$\|\nabla\tilde{\boldsymbol{\phi}}\| \leqslant Ch^{-1}\|\tilde{\boldsymbol{\phi}}\|$$

and Young's inequality to obtain the first claimed inequality.

The second claimed inequality comes from rewriting the first equation of (4.1) in terms of the pressure then using the Cauchy–Schwarz inequality to write

$$\frac{(\tilde{p}, \nabla\cdot\mathbf{v}^h)}{\|\nabla\mathbf{v}\|} \leqslant C\big\{\|p\| + \nu\|\nabla\mathbf{w}\| + \nu\delta^2\|\nabla\boldsymbol{\phi}\| + \nu\|\nabla\tilde{\mathbf{w}}\| + \nu\delta^2\|\tilde{\boldsymbol{\phi}}\|\big\}.$$

The inf-sup condition (2.3) gives the result. □

Proposition 4.2 *(Error in the modified Stokes projection). Suppose the discrete inf-sup condition* (2.3) *holds. Then, $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}}, \tilde{p})$ exists uniquely in $(\mathbf{X}^h, \mathbf{X}^h, Q^h)$ and satisfies*

$$\begin{aligned}&\nu\big\|\nabla(\mathbf{w}-\tilde{\mathbf{w}})\big\|^2 + \nu\delta^2\|\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}\|^2 \\ &\quad\leqslant C\Big\{\inf_{\hat{\mathbf{w}}\in\mathbf{X}^h}\big(\nu+\nu\delta^2h^{-2}\big)\big\|\nabla(\mathbf{w}-\hat{\mathbf{w}})\big\|^2 + \inf_{\hat{\boldsymbol{\phi}}\in\mathbf{X}^h}\nu\delta^4\big(\big\|\nabla(\boldsymbol{\phi}-\hat{\boldsymbol{\phi}})\big\|^2 + h^{-2}\|\boldsymbol{\phi}-\hat{\boldsymbol{\phi}}\|^2\big) \\ &\qquad+ \inf_{q^h\in Q^h}\nu^{-1}\big\|p-q^h\big\|^2\Big\},\end{aligned}$$

where $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}})$ is the modified Stokes projection and C is a constant independent of ν, δ and h.

Proof. Since $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}}, \tilde{p})$ is the solution of a linear system, the a priori bounds in Proposition 4.1 (and the discrete inf-sup condition (2.3)) are enough to guarantee existence and uniqueness.

For the error bound, set $\mathbf{e} = \mathbf{w} - \tilde{\mathbf{w}}$ and decompose the error in two parts as $\mathbf{e} = \boldsymbol{\eta} - \boldsymbol{\psi}^h = (\mathbf{w} - \hat{\mathbf{w}}) - (\tilde{\mathbf{w}} - \hat{\mathbf{w}})$, where $\hat{\mathbf{w}}$ is a approximation of $\mathbf{w} \in \mathbf{V}^h$. Then Eq. (4.2) becomes

$$\begin{aligned} &\nu\big(\nabla\boldsymbol{\psi}^h, \nabla\mathbf{v}^h\big) + \nu\delta^2\big(\nabla\tilde{\boldsymbol{\phi}}, \nabla\mathbf{v}^h\big) \\ &\quad = \nu\big(\nabla\boldsymbol{\eta}, \nabla\mathbf{v}^h\big) + \nu\delta^2\big(\nabla\boldsymbol{\phi}, \nabla\mathbf{v}^h\big) - \big(p - q^h, \nabla\cdot\mathbf{v}^h\big), \end{aligned} \tag{4.6}$$

$$\big(\nabla(\boldsymbol{\eta} - \boldsymbol{\psi}^h), \nabla\boldsymbol{\xi}^h\big) = \big(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}, \boldsymbol{\xi}^h\big). \tag{4.7}$$

We pick $\mathbf{v}^h = \boldsymbol{\psi}^h$ and subtract $\nu\delta^2(\nabla I(\boldsymbol{\phi}), \nabla\boldsymbol{\psi}^h)$ in both sides of (4.6), where $I(\boldsymbol{\phi})$ is the L^2 orthogonal projection of $\boldsymbol{\phi}$ in $\mathbf{X}^h$. Also, in (4.7), we choose $\boldsymbol{\xi}^h = \tilde{\boldsymbol{\phi}} - I(\boldsymbol{\phi})$ and use the orthogonality. This gives

$$\begin{aligned} &\nu\big\|\nabla\boldsymbol{\psi}^h\big\|^2 + \nu\delta^2\big(\nabla\big(\tilde{\boldsymbol{\phi}} - I(\boldsymbol{\phi})\big), \nabla\boldsymbol{\psi}^h\big) \\ &\quad = \nu\big(\nabla\boldsymbol{\eta}, \nabla\boldsymbol{\psi}^h\big) + \nu\delta^2\big(\nabla\big(\boldsymbol{\phi} - I(\boldsymbol{\phi})\big), \nabla\boldsymbol{\psi}^h\big) - \big(p - q^h, \nabla\cdot\boldsymbol{\psi}^h\big), \end{aligned} \tag{4.8}$$

$$\big(\nabla\boldsymbol{\eta}, \nabla\big(\tilde{\boldsymbol{\phi}} - I(\boldsymbol{\phi})\big)\big) + \big\|\tilde{\boldsymbol{\phi}} - I(\boldsymbol{\phi})\big\|^2 = \big(\nabla\boldsymbol{\psi}^h, \nabla\big(\tilde{\boldsymbol{\phi}} - I(\boldsymbol{\phi})\big)\big). \tag{4.9}$$

Multiply (4.9) by $\nu\delta^2$ and substitute the resulting expression in the left-hand side of (4.8). With the application of the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} &\nu\big\|\nabla\boldsymbol{\psi}^h\big\|^2 + \nu\delta^2\big\|\tilde{\boldsymbol{\phi}} - I(\boldsymbol{\phi})\big\|^2 \\ &\quad \leqslant \nu\|\nabla\boldsymbol{\eta}\|\big\|\nabla\boldsymbol{\psi}^h\big\| + \nu\delta^2\|\nabla\boldsymbol{\eta}\|\big\|\nabla\big(\tilde{\boldsymbol{\phi}} - I(\boldsymbol{\phi})\big)\big\| + \nu\delta^2\big\|\nabla\big(\boldsymbol{\phi} - I(\boldsymbol{\phi})\big)\big\|\big\|\nabla\boldsymbol{\psi}^h\big\| \\ &\qquad + \big\|p - q^h\big\|\big\|\nabla\cdot\boldsymbol{\psi}^h\big\|. \end{aligned} \tag{4.10}$$

By using the following inverse inequality,

$$\big\|\nabla\big(\tilde{\boldsymbol{\phi}} - I(\boldsymbol{\phi})\big)\big\| \leqslant Ch^{-1}\big\|\tilde{\boldsymbol{\phi}} - I(\boldsymbol{\phi})\big\|,$$

and using Young's inequality for the terms on the right-hand side of (4.10), we get

$$\begin{aligned} &\nu\big\|\nabla\boldsymbol{\psi}^h\big\|^2 + 2\nu\delta^2\big\|\tilde{\boldsymbol{\phi}} - I(\boldsymbol{\phi})\big\|^2 \\ &\quad \leqslant C\big\{\big(\nu + \nu\delta^2h^{-2}\big)\|\nabla\boldsymbol{\eta}\|^2 + \nu\delta^4\big\|\nabla\big(\boldsymbol{\phi} - I(\boldsymbol{\phi})\big)\big\|^2 + \nu^{-1}\big\|p - q^h\big\|^2\big\}. \end{aligned} \tag{4.11}$$

In (4.11), we need a bound for the term $\|\nabla(\boldsymbol{\phi} - I(\boldsymbol{\phi}))\|$. To get the corresponding bound, we add and subtract terms as

$$\big\|\nabla\big(\boldsymbol{\phi} - I(\boldsymbol{\phi})\big)\big\| \leqslant \big\|\nabla(\boldsymbol{\phi} - \hat{\boldsymbol{\phi}})\big\| + \big\|\nabla\big(\hat{\boldsymbol{\phi}} - I(\boldsymbol{\phi})\big)\big\|, \tag{4.12}$$

where $\hat{\boldsymbol{\phi}}$ is a interpolant of $\boldsymbol{\phi} \in X^h$. Applying the inverse inequality for the last term in (4.12), and adding and subtracting terms gives

$$\big\|\nabla\big(\hat{\boldsymbol{\phi}} - I(\boldsymbol{\phi})\big)\big\| \leqslant Ch^{-1}\big\|\hat{\boldsymbol{\phi}} - I(\boldsymbol{\phi})\big\| \leqslant Ch^{-1}\big(\|\boldsymbol{\phi} - \hat{\boldsymbol{\phi}}\| + \big\|\boldsymbol{\phi} - I(\boldsymbol{\phi})\big\|\big). \tag{4.13}$$

Combining (4.13) and (4.12) yields

$$\big\|\nabla\big(\boldsymbol{\phi} - I(\boldsymbol{\phi})\big)\big\| \leqslant \big\|\nabla(\boldsymbol{\phi} - \hat{\boldsymbol{\phi}})\big\| + Ch^{-1}\big(\|\boldsymbol{\phi} - \hat{\boldsymbol{\phi}}\| + \big\|\boldsymbol{\phi} - I(\boldsymbol{\phi})\big\|\big). \tag{4.14}$$

Since each term on the right-hand side of (4.14) is optimal, we get an optimal error bound for the term $\|\nabla(\boldsymbol{\phi} - I(\boldsymbol{\phi}))\|$.

Final result follows from applying the triangle inequality and taking the infimum over $\mathbf{V}^h$ in (4.11). Note that, under the inf-sup condition and the condition $\nabla\cdot\mathbf{w} = 0$, the infimum over $\mathbf{V}^h$ can be replaced by infimum over $\mathbf{X}^h$ (Girault and Raviart [6]). □

Remark 4.1. The statements of Propositions 4.1 and 4.2 suggest that to get an optimal bound, one has to choose $\delta = \mathcal{O}(h)$.

Remark 4.2. The error in the modified Stokes projection $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}})$ is bounded by approximation theoretic terms, according to Proposition 4.2.

The semi-discrete convergence analysis of the new discretization uses properties of the modified Stokes projection defined above. It follows the usual finite element technique and calls for the use of Gronwall's inequality. A term similar to the nonlinear one that arises in the analysis of the Navier–Stokes equations appears here, making it necessary to make a priori assumptions on $\mathbf{w}$.

Theorem 4.1. *Let* $(\mathbf{w}, p)$ *be the solution of* (3.2)–(3.4). *Under the assumptions that* $\nabla\mathbf{w} \in L^4(0,T;L^2(\Omega))$, $\mathbf{w}_t \in L^2(0,T;H^{-1}(\Omega))$, $\nabla\mathbf{w}_t \in L^2(0,T;L^2(\Omega))$ *and* $p \in L^2(0,T;L_0^2(\Omega))$, *the error* $\mathbf{e} = \mathbf{w} - \mathbf{w}^h$ *satisfies*

$$
\begin{aligned}
&\|\mathbf{e}\|^2_{L^\infty(0,T;L^2(\Omega))} + \delta^2\|\nabla\mathbf{e}\|^2_{L^\infty(0,T;L^2(\Omega))} + \nu\|\nabla\mathbf{e}\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + \nu\delta^2\big\|\boldsymbol{\phi} - \boldsymbol{\phi}^h\big\|^2_{L^2(0,T;L^2(\Omega))} \\
&\leqslant CC^*\big(\big\|\mathbf{w}(0) - \mathbf{w}^h(0)\big\|^2 + \big\|\nabla\big(\mathbf{w}(0) - \mathbf{w}^h(0)\big)\big\|^2\big) + C\mathcal{F}(\mathbf{w} - \tilde{\mathbf{w}}, \boldsymbol{\phi} - \tilde{\boldsymbol{\phi}})
\end{aligned}
$$

where $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}})$ *is the modified Stokes projection,* $C^*(T) = \exp(\frac{C}{\nu^3}\int_0^T \|\nabla\mathbf{w}\|^4\,dt')$ *and*

$$
\begin{aligned}
&\mathcal{F}(\mathbf{w} - \tilde{\mathbf{w}}, \boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}) \\
&\quad = \|\mathbf{w} - \tilde{\mathbf{w}}\|^2 + \delta^2\big\|\nabla(\mathbf{w} - \tilde{\mathbf{w}})\big\|^2_{L^2(0;T;L^2(\Omega))} + \nu\delta^2\|\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}\|^2_{L^2(0;T;L^2(\Omega))} \\
&\qquad + C^*(T)\big[\big\|\mathbf{w}(0) - \tilde{\mathbf{w}}(0)\big\|^2 + \big\|\nabla\big(\mathbf{w}(0) - \tilde{\mathbf{w}}(0)\big)\big\|^2 \\
&\qquad + \nu^{-1}\big\|(\mathbf{w} - \tilde{\mathbf{w}})_t\big\|^2_{L^2(0;T;H^{-1}(\Omega))} + \nu^{-1}\delta^4\big\|\nabla(\mathbf{w} - \tilde{\mathbf{w}})_t\big\|^2_{L^2(0;T;L^2(\Omega))} \\
&\qquad + \big(\big\|\nabla\mathbf{w}^h\big\|_{L^2(0;T;L^2(\Omega))} + \|\nabla\mathbf{w}\|^2_{L^4(0;T;L^2(\Omega))}\big)\big\|\nabla(\mathbf{w} - \tilde{\mathbf{w}})\big\|^2_{L^4(0;T;L^2(\Omega))}\big].
\end{aligned}
$$

Proof. We first set $\mathbf{v} = \mathbf{v}^h$ in (3.2) and $\xi = \xi^h$ in (3.3). Then, subtracting (3.2) from (3.12) and (3.3) from (3.13) and letting $\mathbf{e} = \mathbf{w} - \mathbf{w}^h$ give the following error equations:

$$
\begin{aligned}
&\big(\mathbf{e}_t, \mathbf{v}^h\big) + \delta^2\big(\nabla\mathbf{e}_t, \nabla\mathbf{v}^h\big) + \nu\big(\nabla\mathbf{e}, \nabla\mathbf{v}^h\big) + b\big(\mathbf{w}, \mathbf{w}, \mathbf{v}^h\big) - b\big(\mathbf{w}^h, \mathbf{w}^h, \mathbf{v}^h\big) \\
&\quad + \nu\delta^2\big(\nabla\big(\boldsymbol{\phi} - \boldsymbol{\phi}^h\big), \nabla\mathbf{v}^h\big) - \big(p - q^h, \nabla\cdot\mathbf{v}^h\big) = 0, \quad \forall\mathbf{v}^h \in \mathbf{V}^h, \qquad (4.15)\\
&\big(\nabla\mathbf{e}, \nabla\boldsymbol{\xi}^h\big) = \big(\boldsymbol{\phi} - \boldsymbol{\phi}^h, \boldsymbol{\xi}^h\big), \quad \forall\boldsymbol{\xi}^h \in \mathbf{X}^h. \qquad (4.16)
\end{aligned}
$$

Decompose the error in two parts: $\mathbf{e} = \boldsymbol{\eta} - \boldsymbol{\psi}^h$ where $\boldsymbol{\eta} = \mathbf{w} - \tilde{\mathbf{w}}$, $\boldsymbol{\psi}^h = \mathbf{w}^h - \tilde{\mathbf{w}}$, and add and subtract $\tilde{\boldsymbol{\phi}}$ in (4.15), where $\tilde{\mathbf{w}} \in \mathbf{V}^h$, $\tilde{\boldsymbol{\phi}} \in \mathbf{X}^h$ are chosen as the modified Stokes projection, defined via (4.2)–(4.3). Putting all these together and setting $\mathbf{v}^h = \boldsymbol{\psi}^h$ in (4.15) and $\boldsymbol{\xi} = \boldsymbol{\phi}^h - \tilde{\boldsymbol{\phi}}$ in (4.16) yield

$$
\begin{aligned}
&\big(\boldsymbol{\psi}^h_t, \boldsymbol{\psi}^h\big) + \delta^2\big(\nabla\boldsymbol{\psi}^h_t, \nabla\boldsymbol{\psi}^h\big) + \nu\big(\nabla\boldsymbol{\psi}^h, \nabla\boldsymbol{\psi}^h\big) + \nu\delta^2\big(\nabla\big(\boldsymbol{\phi}^h - \tilde{\boldsymbol{\phi}}\big), \nabla\boldsymbol{\psi}^h\big) \\
&\quad = \big(\boldsymbol{\eta}_t, \boldsymbol{\psi}^h\big) + \delta^2\big(\nabla\boldsymbol{\eta}_t, \nabla\boldsymbol{\psi}^h\big) + b\big(\mathbf{w}, \mathbf{w}, \mathbf{v}^h\big) - b\big(\mathbf{w}^h, \mathbf{w}^h, \mathbf{v}^h\big), \qquad (4.17)\\
&\big(\nabla\boldsymbol{\psi}^h, \nabla\big(\boldsymbol{\phi}^h - \tilde{\boldsymbol{\phi}}\big)\big) = \big(\boldsymbol{\phi}^h - \tilde{\boldsymbol{\phi}}, \boldsymbol{\phi}^h - \tilde{\boldsymbol{\phi}}\big). \qquad (4.18)
\end{aligned}
$$

Note that, since $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}})$ is taken to be the modified Stokes projection of $(\mathbf{w}, \boldsymbol{\phi})$ in $(\mathbf{V}^h, \mathbf{X}^h)$, some of the terms in the error equation (4.17) vanish.

We then multiply both sides of (4.18) by $\nu\delta^2$, substitute in (4.17), and apply Cauchy–Schwarz inequality. This gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{\psi}^h\right\|^2 + \frac{\delta^2}{2}\frac{d}{dt}\left\|\nabla\boldsymbol{\psi}^h\right\|^2 + \nu\left\|\nabla\boldsymbol{\psi}^h\right\|^2 + \nu\delta^2\left\|\boldsymbol{\phi}^h - \tilde{\boldsymbol{\phi}}\right\|^2 \\
&\quad \leqslant \|\boldsymbol{\eta}_t\|_{-1}\left\|\nabla\boldsymbol{\psi}^h\right\| + \delta^2\|\nabla\boldsymbol{\eta}_t\|\left\|\nabla\boldsymbol{\psi}^h\right\| + \left|b\left(\mathbf{w}, \mathbf{w}, \boldsymbol{\psi}^h\right) - b\left(\mathbf{w}^h, \mathbf{w}^h, \boldsymbol{\psi}^h\right)\right|. \qquad (4.19)
\end{aligned}$$

The nonlinear term on the right-hand side of (4.19) is decomposed into three parts. This reduces to

$$b\left(\mathbf{w}, \mathbf{w}, \boldsymbol{\psi}^h\right) - b\left(\mathbf{w}^h, \mathbf{w}^h, \boldsymbol{\psi}^h\right) = b\left(\boldsymbol{\eta}, \mathbf{w}, \boldsymbol{\psi}^h\right) - b\left(\boldsymbol{\psi}^h, \mathbf{w}, \boldsymbol{\psi}^h\right) + b\left(\mathbf{w}^h, \boldsymbol{\eta}, \boldsymbol{\psi}^h\right).$$

By applying the improved estimate (2.2), Poincaré's and Young's inequalities, the nonlinear terms are bounded as

$$\begin{aligned}
b\left(\boldsymbol{\eta}, \mathbf{w}, \boldsymbol{\psi}^h\right) &\leqslant C\|\boldsymbol{\eta}\|^{1/2}\|\nabla\boldsymbol{\eta}\|^{1/2}\|\nabla\mathbf{w}\|\left\|\nabla\boldsymbol{\psi}^h\right\| \\
&\leqslant \frac{\epsilon}{4}\left\|\nabla\boldsymbol{\psi}^h\right\|^2 + \frac{C}{\epsilon}\|\nabla\boldsymbol{\eta}\|^2\|\nabla\mathbf{w}\|^2, \\
b\left(\boldsymbol{\psi}^h, \mathbf{w}, \boldsymbol{\psi}^h\right) &\leqslant \left\|\nabla\boldsymbol{\psi}^h\right\|^{3/2}\left\|\boldsymbol{\psi}^h\right\|^{1/2}\|\nabla\mathbf{w}\| \\
&\leqslant \frac{\epsilon}{2}\left\|\nabla\boldsymbol{\psi}^h\right\|^2 + \frac{C}{\epsilon^3}\|\nabla\mathbf{w}\|^4\left\|\boldsymbol{\psi}^h\right\|^2, \\
b\left(\mathbf{w}^h, \boldsymbol{\eta}, \boldsymbol{\psi}^h\right) &\leqslant C\left\|\mathbf{w}^h\right\|^{1/2}\left\|\nabla\mathbf{w}^h\right\|^{1/2}\|\nabla\boldsymbol{\eta}\|\left\|\nabla\boldsymbol{\psi}^h\right\| \\
&\leqslant \frac{\epsilon}{4}\left\|\nabla\boldsymbol{\psi}^h\right\|^2 + \frac{C}{\epsilon}\left\|\mathbf{w}^h\right\|\left\|\nabla\mathbf{w}^h\right\|\|\nabla\boldsymbol{\eta}\|^2.
\end{aligned}$$

On the right-hand side of (4.19), we apply Young's inequality and choose $\epsilon = \nu/4$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{\psi}^h\right\|^2 + \frac{\delta^2}{2}\frac{d}{dt}\left\|\nabla\boldsymbol{\psi}^h\right\|^2 + \frac{\nu}{2}\left\|\nabla\boldsymbol{\psi}^h\right\|^2 + \nu\delta^2\left\|\boldsymbol{\phi}^h - \tilde{\boldsymbol{\phi}}\right\|^2 \\
&\quad \leqslant 2\nu^{-1}\|\boldsymbol{\eta}_t\|_{-1}^2 + \nu^{-1}\delta^4\|\nabla\boldsymbol{\eta}_t\|^2 + \frac{C}{\nu}\left(\|\nabla\mathbf{w}\|^2 + \left\|\mathbf{w}^h\right\|\left\|\nabla\mathbf{w}^h\right\|\right)\|\nabla\boldsymbol{\eta}\|^2 \\
&\qquad + \frac{C}{\nu^3}\|\nabla\mathbf{w}\|^4\left\|\boldsymbol{\psi}^h\right\|^2.
\end{aligned}$$

Since by assumption $\|\nabla\mathbf{w}\|^4 \in L^1(0, T)$, Gronwall's inequality implies that

$$\begin{aligned}
&\left\|\boldsymbol{\psi}^h\right\|^2 + \delta^2\left\|\nabla\boldsymbol{\psi}^h\right\|^2 + \int_0^t \left[\nu\left\|\nabla\boldsymbol{\psi}^h\right\|^2 + 2\nu\delta^2\left\|\boldsymbol{\phi}^h - \tilde{\boldsymbol{\phi}}\right\|^2\right] dt' \\
&\quad \leqslant C^*(t)\left(\left\|\boldsymbol{\psi}^h(0)\right\|^2 + \left\|\nabla\boldsymbol{\psi}^h(0)\right\|^2\right) + CC^*(t)\int_0^t \Bigg[\nu^{-1}\|\boldsymbol{\eta}_t\|_{-1}^2 + \nu^{-1}\delta^4\|\nabla\boldsymbol{\eta}_t\|^2 \\
&\qquad + \frac{1}{\nu}\left(\|\nabla\mathbf{w}\|^2 + \left\|\mathbf{w}^h\right\|\left\|\nabla\mathbf{w}^h\right\|\right)\|\nabla\boldsymbol{\eta}\|^2\Bigg] dt',
\end{aligned}$$

where $C^*(t)=\exp(\frac{C}{\nu^3}\int_0^t\|\nabla\mathbf{w}\|^4\,dt')$. In order to complete proof, one has to study the $L^1(0,T)$ regularity of terms in the last inequality. Note that we can bound by using Cauchy–Schwarz inequality

$$\int_0^t \|\nabla\mathbf{w}\|^2\|\nabla\boldsymbol{\eta}\|^2\,dt' \leqslant \|\nabla\mathbf{w}\|^2_{L^4(0,t;L^2(\Omega))}\|\nabla\boldsymbol{\eta}\|^2_{L^4(0,t;L^2(\Omega))} < \infty.$$

Similarly, using Hölder's inequality and Lemma 3.3 imply that

$$\begin{aligned}
&\int_0^T \|\mathbf{w}^h\|\,\|\nabla\mathbf{w}^h\|\,\|\nabla\boldsymbol{\eta}\|^2\,dt' \\
&\quad\leqslant \|\mathbf{w}^h\|_{L^\infty(0,t;L^2(\Omega))}\|\nabla\mathbf{w}^h\|_{L^2(0,t;L^2(\Omega))}\|\nabla\boldsymbol{\eta}\|^2_{L^4(0,t;L^2(\Omega))} \\
&\quad\leqslant C\left(\frac{1}{\nu^{1/2}}\|\mathbf{w}^h(0)\|^2+\frac{\delta^2}{\nu^{1/2}}\|\nabla\mathbf{w}^h(0)\|^2+\frac{1}{\nu^{3/2}}\|\mathbf{f}\|^2_{L^2(0,t;H^{-1}(\Omega))}\right)\|\nabla\boldsymbol{\eta}\|^2_{L^4(0,t;L^2(\Omega))} \\
&\quad<\infty.
\end{aligned}$$

The stated error estimate now follows by applying the triangle inequality. □

5. Time averaged errors

In this section, we analyze time averaged errors. In practical flow computations, pointwise flow quantities may not make sense, whereas statistics of flow quantities do. The analysis we employ here follows the same idea as the one in [10], where statistics of weak solutions of the Navier–Stokes are investigated. Accordingly, we put ourselves in the case where the solution to the time dependent problem converges to a stationary solution, provided the steady-state body force is small enough. In this context, statistics are optimally computable. In the general case, it is not known if a closed estimate is feasible (see [10]).

We must point out that weak solutions of the Zeroth Order Model are indeed strong solutions and satisfy an energy equality [12]. This has been proven for the periodic case, but it is reasonable to assume that the same holds true in the nonperiodic case (operationally, this will allow us to choose, e.g., $\mathbf{w}$ as a test function in the estimates below).

We will need properties of the steady-state solution, denoted with superscript $*$, so we first consider the equilibrium variational formulation of problem (1.4) when $\mathbf{f}(x,t)\to\mathbf{f}^*(x)$ as $t\to\infty$: Find $(\mathbf{w}^*,\boldsymbol{\phi}^*,p^*)\in(\mathbf{X},\mathbf{X},Q)$ such that

$$\nu(\nabla\mathbf{w}^*,\nabla\mathbf{v})+\nu\delta^2(\nabla\boldsymbol{\phi}^*,\nabla\mathbf{v})+b(\mathbf{w}^*,\mathbf{w}^*,\mathbf{v})-(p^*,\nabla\cdot\mathbf{v})=(\mathbf{f}^*,\mathbf{v}),\quad \forall\mathbf{v}\in\mathbf{X},\tag{5.1}$$

$$(\nabla\mathbf{w}^*,\nabla\boldsymbol{\xi})=(\boldsymbol{\phi}^*,\boldsymbol{\xi}),\quad \forall\boldsymbol{\xi}\in\mathbf{X},\tag{5.2}$$

$$(q,\nabla\cdot\mathbf{w}^*)=0,\quad \forall q\in Q.\tag{5.3}$$

In $\mathbf{V}$, the variational formulation becomes: Find $(\mathbf{w}^*,\boldsymbol{\phi}^*)\in(\mathbf{V},\mathbf{X})$ satisfying

$$\nu(\nabla\mathbf{w}^*,\nabla\mathbf{v})+\nu\delta^2(\nabla\boldsymbol{\phi}^*,\nabla\mathbf{v})+b(\mathbf{w}^*,\mathbf{w}^*,\mathbf{v})=(\mathbf{f}^*,\mathbf{v}),\quad \forall\mathbf{v}\in\mathbf{V},\tag{5.4}$$

$$(\nabla\mathbf{w}^*,\nabla\boldsymbol{\xi})=(\boldsymbol{\phi}^*,\boldsymbol{\xi}),\quad \forall\boldsymbol{\xi}\in\mathbf{X}.\tag{5.5}$$

Our first result in this section gives an a priori bound on $(\mathbf{w}^*,\boldsymbol{\phi}^*)$.

Lemma 5.1. *The solution* $(\mathbf{w}^*, \boldsymbol{\phi}^*)$ *to* (5.4)–(5.5) *is bounded such that*

$$\|\nabla \mathbf{w}^*\|^2 + 2\delta^2\|\boldsymbol{\phi}^*\|^2 \leqslant \nu^{-2}\|\mathbf{f}^*\|_{-1}^2.$$

Proof. Setting $\mathbf{v} = \mathbf{w}^*$ in (5.4) and $\boldsymbol{\xi} = \boldsymbol{\phi}^*$ in (5.5) gives the claimed result. □

This result, together with assumptions on the steady-state body force $\mathbf{f}^*$ and on its relationship with the time dependent body force $\mathbf{f}$ gives, in turn, a relationship between the solutions $\mathbf{w}$ and $\mathbf{w}^*$.

Proposition 5.1. *Let* $\mathbf{f} \in L^\infty(0,\infty; H^{-1}(\Omega))$. *Suppose that for all* T *sufficiently large,* $\|\mathbf{f} - \mathbf{f}^*\|_{-1}$ *is bounded for* $0 \leqslant t \leqslant T/2$ *and* $\int_{T/2}^T \|\mathbf{f}(.,t) - \mathbf{f}^*(.)\|_{-1}^2\,dt \to 0$ *as* $T \to \infty$ *then* $\mathbf{w}(x,t) \to \mathbf{w}^*(x)$ *in* $H^1(\Omega)$, *whenever* $M\nu^{-2}\|\mathbf{f}^*\|_{-1} := \alpha < 1$.

Proof. The idea behind this proof is to divide the time axis in two parts. The first, where the difference between $\mathbf{f}$ and $\mathbf{f}^*$ is bounded (and the exponentials involved tend to zero), and the second part, which becomes small when $\mathbf{f}$ and $\mathbf{f}^*$ are sufficiently close.

We first subtract (5.4) from (3.5) and (5.5) from (3.6) and set $\mathbf{W} = \mathbf{w} - \mathbf{w}^*$ and $\boldsymbol{\Phi} = \boldsymbol{\phi} - \boldsymbol{\phi}^*$. Then, we have an equation of the form

$$\begin{aligned}&(\mathbf{W}_t, \mathbf{v}) + \delta^2(\nabla \mathbf{W}_t, \nabla \mathbf{v}) + \nu(\nabla \mathbf{W}, \nabla \mathbf{v}) + \nu\delta^2(\nabla \boldsymbol{\Phi}, \nabla \mathbf{v}) + b(\mathbf{w}, \mathbf{w}, \mathbf{v})\\ &\quad - b(\mathbf{w}^*, \mathbf{w}^*, \mathbf{v}) = (\mathbf{f} - \mathbf{f}^*, \mathbf{v}), \end{aligned} \tag{5.6}$$

$$(\nabla \mathbf{W}, \nabla \boldsymbol{\xi}) = (\boldsymbol{\Phi}, \boldsymbol{\xi}), \tag{5.7}$$

for all $(\mathbf{v}, \boldsymbol{\xi}) \in (\mathbf{V}, \mathbf{X})$.

Setting $\mathbf{v} = \mathbf{W}$ in (5.6) and $\boldsymbol{\xi} = \boldsymbol{\Phi}$ in (5.7), adding the two resulting equations together, adding and subtracting the term $b(\mathbf{w}, \mathbf{w}^*, \mathbf{W})$ and using the skew-symmetry of trilinear form, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\mathbf{W}\|^2 + \delta^2\|\nabla \mathbf{W}\|^2\big) + \nu\big(\|\nabla \mathbf{W}\|^2 + \delta^2\|\boldsymbol{\Phi}\|^2\big)\\ &\quad = -b(\mathbf{W}, \mathbf{w}^*, \mathbf{W}) + (\mathbf{f} - \mathbf{f}^*, \mathbf{W}). \end{aligned} \tag{5.8}$$

Using the bound on nonlinear term, $b(\mathbf{W}, \mathbf{w}^*, \mathbf{W}) \leqslant M\|\nabla \mathbf{w}^*\|\,\|\nabla \mathbf{W}\|^2$, together with the a priori bound $\|\nabla \mathbf{w}^*\| \leqslant \nu^{-1}\|\mathbf{f}^*\|_{-1}$ (from Lemma 5.1), followed by $(\mathbf{f} - \mathbf{f}^*, \mathbf{W}) \leqslant \|\mathbf{f} - \mathbf{f}^*\|_{-1}\|\nabla \mathbf{W}\|$ and Young's inequality, we get, for fixed $\epsilon > 0$,

$$\frac{1}{2}\frac{d}{dt}\big(\|\mathbf{W}\|^2 + \delta^2\|\nabla \mathbf{W}\|^2\big) + \nu(1 - \alpha - \epsilon)\|\nabla \mathbf{W}\|^2 + \nu\delta^2\|\boldsymbol{\Phi}\|^2 \leqslant \frac{1}{4\epsilon\nu}\|\mathbf{f} - \mathbf{f}^*\|_{-1}^2. \tag{5.9}$$

Letting $\boldsymbol{\xi} = \mathbf{W}$ in Eq. (5.7) and using Poincaré's inequality, we find that $\|\nabla \mathbf{W}\| \leqslant C_P\|\boldsymbol{\Phi}\|$. Application of Poincaré's inequality to (5.9) yields

$$\frac{d}{dt}\big(\|\mathbf{W}\|^2 + \delta^2\|\nabla \mathbf{W}\|^2\big) + 2C_P^{-2}\nu\big((1 - \alpha - \epsilon)\|\mathbf{W}\|^2 + \delta^2\|\nabla \mathbf{W}\|^2\big) \leqslant \frac{1}{2\epsilon\nu}\|\mathbf{f} - \mathbf{f}^*\|_{-1}^2.$$

Set $y = \|\mathbf{W}\|^2 + \delta^2\|\nabla \mathbf{W}\|^2$. Then, for ϵ small enough, $K := 2\nu C_P^{-2}(1 - \alpha - \epsilon) > 0$ and this inequality becomes

$$\frac{dy}{dt} + Ky < \frac{1}{2\epsilon\nu}\|\mathbf{f} - \mathbf{f}^*\|_{-1}^2. \tag{5.10}$$

Choosing an integrating factor, Eq. (5.10) gives

$$y(T) \leqslant y(0)e^{-KT} + \frac{1}{2\epsilon\nu}\int_0^{\frac{T}{2}} e^{K(t-T)}\|\mathbf{f}-\mathbf{f}^*\|_{-1}^2\,dt + \frac{1}{2\epsilon\nu}\int_{\frac{T}{2}}^{T} e^{K(t-T)}\|\mathbf{f}-\mathbf{f}^*\|_{-1}^2\,dt. \tag{5.11}$$

The first integral on the right-hand side of (5.11) can be estimated:

$$\int_0^{\frac{T}{2}} e^{K(t-T)}\|\mathbf{f}-\mathbf{f}^*\|_{-1}^2\,dt \leqslant K^{-1}\big(e^{-K\frac{T}{2}} - e^{-KT}\big)\|\mathbf{f}-\mathbf{f}^*\|_{L^\infty(0,T/2;H^{-1}(\Omega))}^2,$$

and the second integral can be estimated:

$$\int_{\frac{T}{2}}^{T} e^{K(t-T)}\|\mathbf{f}-\mathbf{f}^*\|_{-1}^2\,dt \leqslant \int_{\frac{T}{2}}^{T} \|\mathbf{f}-\mathbf{f}^*\|_{-1}^2\,dt,$$

since $e^{K(t-T)} \leqslant 1$ for $\frac{T}{2} \leqslant t \leqslant T$.

Combining everything together, (5.11) becomes

$$y(T) \leqslant y(0)e^{-KT} + \frac{1}{2\epsilon\nu K}\big(e^{-K\frac{T}{2}} - e^{-KT}\big)\|\mathbf{f}-\mathbf{f}^*\|_{L^\infty(0,T/2;H^{-1}(\Omega))}^2 + \frac{1}{2\epsilon\nu}\|\mathbf{f}-\mathbf{f}^*\|_{L^2(T/2,T;H^{-1}(\Omega))}^2.$$

Letting $T \to \infty$ concludes the proof. □

The first result involving time averages shows that the time averaged energy dissipation rate of the nonstationary solution converges, as $t \to \infty$, to the steady-state energy dissipation rate.

Proposition 5.2. *Under the same assumptions of Proposition* 5.1*, we can show that*

$$\big\langle \varepsilon(\mathbf{w}-\mathbf{w}^*, \boldsymbol{\phi}-\boldsymbol{\phi}^*)\big\rangle = 0.$$

Proof. The proof is similar to the proof of Proposition 5.1, so we start directly from Eq. (5.9), with $\mathbf{W} = \mathbf{w} - \mathbf{w}^*$ and $\boldsymbol{\Phi} = \boldsymbol{\phi} - \boldsymbol{\phi}^*$. We multiply it by 2, use the fact that $\min\{1-\alpha-\epsilon, 1\} = 1-\alpha-\epsilon$ and integrate from 0 to T to get

$$\big\|\mathbf{W}(T)\big\|^2 + \delta^2\big\|\nabla\mathbf{W}(T)\big\|^2 + (1-\alpha-\epsilon)\nu\int_0^T \big(\|\nabla\mathbf{W}\|^2 + \delta^2\|\boldsymbol{\Phi}\|^2\big)\,dt \leqslant \big\|\mathbf{W}(0)\big\|^2 + \delta^2\big\|\nabla\mathbf{W}(0)\big\|^2 + \frac{1}{2\epsilon\nu}\int_0^T \|\mathbf{f}-\mathbf{f}^*\|_{-1}^2\,dt. \tag{5.12}$$

Dividing everything by T and taking limit supremum on both sides, we see that the first and second terms on the left-hand side vanish (as a consequence of Lemma 3.1 and of the fact that

$\mathbf{w}^*$ does not depend on time). Observing that $\|\mathbf{W}(0)\|^2 + \delta^2\|\nabla \mathbf{W}(0)\|^2$ is a constant and using the hypothesis on $\mathbf{f}$ and $\mathbf{f}^*$, the right-hand side also vanishes and we are left with

$$(1-\alpha-\epsilon)\nu\left\langle \|\nabla \mathbf{W}\|^2 + \delta^2\|\boldsymbol{\Phi}\|^2\right\rangle \leqslant 0.$$

The fact that $(1-\alpha-\epsilon) > 0$ gives the desired result. □

Properties of the approximate solution $\mathbf{w}^{*h}$ are also needed. Thus, we also consider finite element approximation of (5.1)–(5.3). Finite element approximation of the equilibrium solution is to: Find $(\mathbf{w}^{*h}, \boldsymbol{\phi}^{*h}, p^{*h}) \in (\mathbf{X}^h, \mathbf{X}^h, Q^h)$ satisfying

$$\nu\left(\nabla \mathbf{w}^{*h}, \nabla \mathbf{v}^h\right) + \nu\delta^2\left(\nabla \boldsymbol{\phi}^{*h}, \nabla \mathbf{v}\right) + b\left(\mathbf{w}^{*h}, \mathbf{w}^{*h}, \mathbf{v}^h\right) - \left(p^{*h}, \nabla \cdot \mathbf{v}^h\right) = \left(\mathbf{f}^*, \mathbf{v}^h\right), \tag{5.13}$$

$$\left(\nabla \mathbf{w}^{*h}, \nabla \boldsymbol{\xi}^h\right) = \left(\boldsymbol{\phi}^{*h}, \boldsymbol{\xi}^h\right), \tag{5.14}$$

$$\left(q^h, \nabla \cdot \mathbf{w}^{*h}\right) = 0, \tag{5.15}$$

for all $(\mathbf{v}^h, \boldsymbol{\xi}^h, q^h) \in (\mathbf{X}^h, \mathbf{X}^h, Q^h)$, with the usual extension to the formulation in $(\mathbf{V}^h, \mathbf{X}^h)$.

Lemma 5.2. $(\mathbf{w}^{*h}, \boldsymbol{\phi}^{*h})$ *satisfies*

$$\left\|\nabla \mathbf{w}^{*h}\right\|^2 + 2\delta^2\left\|\boldsymbol{\phi}^{*h}\right\|^2 \leqslant \nu^{-2}\|\mathbf{f}^*\|_{-1}^2.$$

Proof. The claim exactly follows the proof of Lemma 5.1. □

We now derive error estimates. The following result uses the modified Stokes projection defined via (4.2)–(4.3). Recall that according to Proposition 4.2, the error in the modified Stokes projection $(\tilde{\mathbf{w}}, \tilde{\boldsymbol{\phi}})$ is bounded.

Proposition 5.3. *Assume that* $(\mathbf{X}^h, Q^h)$ *satisfy an inf-sup condition. Under the small data condition,* $M\nu^{-2}\|\mathbf{f}\|_{-1} := \alpha < 1$*, the following error estimate holds*:

$$\begin{aligned}
&\nu\left\|\nabla\left(\mathbf{w}^* - \mathbf{w}^{*h}\right)\right\|^2 + \nu\delta^2\left\|\boldsymbol{\phi}^* - \boldsymbol{\phi}^{*h}\right\|^2 \\
&\quad\leqslant C\left\{\left(\nu + \nu^{-3}\|\mathbf{f}^*\|_{-1}^2\right)\left\|\nabla(\mathbf{w}^* - \tilde{\mathbf{w}})\right\|^2 + \nu\delta^2\|\boldsymbol{\phi}^* - \tilde{\boldsymbol{\phi}}\|^2\right\}.
\end{aligned}$$

Proof. Subtracting (5.13) from (5.1), for $\mathbf{v}^h \in \mathbf{V}^h$, and subtracting (5.14) from (5.2), for $\boldsymbol{\xi}^h \in \mathbf{X}^h$, we find the error equations:

$$\begin{aligned}
&\nu\left(\nabla\left(\mathbf{w}^* - \mathbf{w}^{*h}\right), \nabla \mathbf{v}^h\right) + \nu\delta^2\left(\nabla\left(\boldsymbol{\phi}^* - \boldsymbol{\phi}^{*h}\right), \nabla \mathbf{v}^h\right) + b\left(\mathbf{w}^*, \mathbf{w}^*, \mathbf{v}^h\right) \\
&\quad - b\left(\mathbf{w}^{*h}, \mathbf{w}^{*h}, \mathbf{v}^h\right) - \left(p^* - q^h, \nabla \cdot \mathbf{v}^h\right) = 0, \\
&\left(\nabla\left(\mathbf{w}^* - \mathbf{w}^{*h}\right), \nabla \boldsymbol{\xi}^h\right) = \left(\boldsymbol{\phi}^* - \boldsymbol{\phi}^{*h}, \boldsymbol{\xi}^h\right).
\end{aligned}$$

Decompose the error as $\mathbf{e} = \boldsymbol{\eta} - \boldsymbol{\psi}^h$, where $\boldsymbol{\eta} = \mathbf{w}^* - \tilde{\mathbf{w}}$, $\boldsymbol{\psi}^h = \mathbf{w}^{*h} - \tilde{\mathbf{w}}$, and add and subtract $\tilde{\boldsymbol{\phi}}$, where $\tilde{\mathbf{w}} \in \mathbf{V}^h, \tilde{\boldsymbol{\phi}} \in \mathbf{X}^h$ are chosen as the modified Stokes projection. Putting all these together and setting $\mathbf{v}^h = \boldsymbol{\psi}^h$ and $\boldsymbol{\xi}^h = \boldsymbol{\phi}^{*h} - \tilde{\boldsymbol{\phi}}$ yields

$$\nu\left\|\nabla \boldsymbol{\psi}^h\right\|^2 + \nu\delta^2\left\|\boldsymbol{\phi}^{*h} - \tilde{\boldsymbol{\phi}}\right\|^2 = b\left(\boldsymbol{\eta}, \mathbf{w}^*, \boldsymbol{\psi}^h\right) - b\left(\boldsymbol{\psi}^h, \mathbf{w}^*, \boldsymbol{\psi}^h\right) + b\left(\mathbf{w}^{*h}, \boldsymbol{\eta}, \boldsymbol{\psi}^h\right), \tag{5.16}$$

where the nonlinear term was decomposed into three parts (by adding and subtracting appropriate terms).

Using the bounds on the trilinear form followed by Young's inequality and the Cauchy–Schwarz inequality, together with the a priori estimates for $\mathbf{w}^*$ and $\mathbf{w}^{*h}$, we write

$$\begin{aligned}
b(\boldsymbol{\eta}, \mathbf{w}^*, \boldsymbol{\psi}^h) &\leqslant M\|\nabla\boldsymbol{\eta}\|\|\nabla\mathbf{w}^*\|\|\nabla\boldsymbol{\psi}^h\| \\
&\leqslant \frac{\nu}{4}\|\nabla\boldsymbol{\psi}^h\|^2 + C\nu^{-1}\|\nabla\mathbf{w}^*\|^2\|\nabla\boldsymbol{\eta}\|^2 \\
&\leqslant \frac{\nu}{4}\|\nabla\boldsymbol{\psi}^h\|^2 + C\nu^{-3}\|\mathbf{f}^*\|_{-1}^2\|\nabla\boldsymbol{\eta}\|^2, \\
b(\boldsymbol{\psi}^h, \mathbf{w}^*, \boldsymbol{\psi}^h) &\leqslant M\|\nabla\mathbf{w}^*\|\|\nabla\boldsymbol{\psi}^h\|^2 \\
&\leqslant M\nu^{-1}\|\mathbf{f}^*\|_{-1}\|\nabla\boldsymbol{\psi}^h\|^2, \\
b(\mathbf{w}^{*h}, \boldsymbol{\eta}, \boldsymbol{\psi}^h) &\leqslant M\|\nabla\boldsymbol{\eta}\|\|\nabla\mathbf{w}^{*h}\|\|\nabla\boldsymbol{\psi}^h\| \\
&\leqslant \frac{\nu}{4}\|\nabla\boldsymbol{\psi}^h\|^2 + C\nu^{-3}\|\mathbf{f}^*\|_{-1}^2\|\nabla\boldsymbol{\eta}\|^2.
\end{aligned}$$

With the help of the estimates above and the fact that $1 - \alpha > 0$, Eq. (5.16) becomes

$$\nu\|\nabla\boldsymbol{\psi}^h\|^2 + \nu\delta^2\|\boldsymbol{\phi}^{*h} - \tilde{\boldsymbol{\phi}}\|^2 \leqslant C\nu^{-3}\|\mathbf{f}^*\|_{-1}^2\|\nabla\boldsymbol{\eta}\|^2.$$

The triangle inequality gives the result. □

The discrete counterpart of Proposition 5.2 is given in the following statement.

Proposition 5.4. *With the same assumptions as in Proposition* 5.2*, we show that*

$$\langle\varepsilon(\mathbf{w}^h - \mathbf{w}^{*h}, \boldsymbol{\phi}^h - \boldsymbol{\phi}^{*h})\rangle = 0.$$

Proof. The argument is the same as in the proof of Proposition 5.2, for the discrete case. □

The next theorem is the major result in this section. It shows that under the condition that the body force driving the flow when it has reached the steady state is small enough, statistics can be accurately computed.

Theorem 5.1. *Assuming the hypotheses of Proposition* 5.1 *hold, then*

$$\langle\varepsilon(\mathbf{w} - \mathbf{w}^h, \boldsymbol{\phi} - \boldsymbol{\phi}^h)\rangle \leqslant C\nu\left(\|\nabla(\mathbf{w}^* - \mathbf{w}^{*h})\|^2 + \delta^2\|\boldsymbol{\phi}^* - \boldsymbol{\phi}^{*h}\|^2\right). \tag{5.17}$$

Proof. Add and subtract $\mathbf{w}^*$, $\mathbf{w}^{*h}$, $\boldsymbol{\phi}^*$, $\boldsymbol{\phi}^{*h}$ appropriately to the formula of $\langle\varepsilon(\mathbf{w} - \mathbf{w}^h, \boldsymbol{\phi} - \boldsymbol{\phi}^h)\rangle$. Then, the proof follows by the application of triangle inequality and from Propositions 5.2 and 5.4. □

Corollary 5.1. *Suppose that the small data condition holds and* $(\mathbf{X}^h, Q^h)$ *satisfy an inf-sup condition. Then,*

$$\langle\varepsilon(\mathbf{w} - \mathbf{w}^h, \boldsymbol{\phi} - \boldsymbol{\phi}^h)\rangle \leqslant C\left\{(\nu + \nu^{-3}\|\mathbf{f}^*\|_{-1}^2)\|\nabla(\mathbf{w}^* - \tilde{\mathbf{w}}^*)\|^2 + \nu\delta^2\|\boldsymbol{\phi}^* - \tilde{\boldsymbol{\phi}}\|^2\right\}.$$

Proof. Use the estimates of Proposition 5.3 on the right-hand side of (5.17). □

6. Conclusions

We proposed a discretization to the Zeroth Order Model based on a mixed variational formulation that represents the natural energy properties of the model well. Optimal convergence rates are obtained if $\delta = \mathcal{O}(h)$, which is consistent with the literature and simulations with other models [8,9]. Additionally, time averaged error estimates are presented. For the special case of asymptotically small body force, they prove to be optimally computable.

References

[1] N.A. Adams, S. Stolz, On the approximate deconvolution procedure for LES, Phys. Fluids 2 (1999) 1699–1701.

[2] N.A. Adams, S. Stolz, Deconvolution methods for subgrid-scale approximation in large eddy simulation, in: B.J. Geurts (Ed.), Modern Simul. Strategies for Turbulent Flow, Edwards, Philadelphia, 2001, pp. 21–44.

[3] L.C. Berselli, T. Iliescu, W.J. Layton, Mathematics of Large Eddy Simulation of Turbulent Flows, Sci. Comput., Springer, 2006.

[4] A. Dunca, Y. Epshteyn, On the Stolz–Adams deconvolution LES model, SIAM J. Math. Anal. 37 (2006) 1890–1902.

[5] J.H. Ferziger, Direct and large eddy simulation of turbulence, in: A. Vincent (Ed.), Numerical Methods in Fluid Mechanics (Centre de Recherches Mathematiques, Universite de Montreal), in: CRM Proc. Lecture Notes, vol. 16, Amer. Math. Soc., 1998.

[6] V. Girault, P.A. Raviart, Finite Element Approximation of the Navier–Stokes Equations, Springer-Verlag, Berlin, 1979.

[7] M. Gunzburger, Finite Element Methods for Viscous Incompressible Flow: A Guide to Theory, Practice, and Algorithms, Academic Press, Boston, 1989.

[8] T. Iliescu, V. John, W.J. Layton, G. Matthies, L. Tobiska, A numerical study of a class of LES models, Int. J. Comput. Fluid Dyn. 17 (2003) 75–85.

[9] V. John, Large Eddy Simulation of Turbulent Incompressible Flows. Analytical and Numerical Results for a Class of LES Models, Lect. Notes Comput. Sci. Eng., vol. 34, Springer-Verlag, Berlin, 2004.

[10] V. John, W. Layton, C.C. Manica, Convergence of time averaged statistics of finite element approximations of the Navier–Stokes equations, 2006, submitted for publication.

[11] W.J. Layton, R. Lewandowski, A simple and stable scale similarity model for large eddy simulation: Energy balance and existence of weak solutions, Appl. Math. Lett. 16 (2003) 1205–1209.

[12] W.J. Layton, R. Lewandowski, On a well-posed turbulence model, Discrete Contin. Dyn. Syst. Ser. B 6 (1) (2006) 111–128.

[13] R. Lewandowski, W. Layton, An exposition on the time-averaged accuracy of approximate-deconvolution, large eddy simulation models of turbulence, Tech. Report, University of Pittsburgh, 2005.

[14] R. Lewandowski, W. Layton, Residual stress in approximate deconvolution models of turbulence, J. Turbul. (2006), in press.

[15] C.C. Manica, S. Kaya Merdan, Convergence analysis of the finite element method for a fundamental model in turbulence, 2006, submitted for publication.

[16] R. Moser, J. Kim, N. Mansour, Direct numerical simulation of turbulent channel flow up to $Re = 590$, Phys. Fluids 11 (1998) 943–945.